From Colors to Numbers

Discovering the hidden structure of algebra

Name: ___________________________ Date: ________________

Part 1: Why Subtraction and Division Cause Problems

■ The Secret

What we're about to show you is usually saved for 7th grade. But after what you discovered with Cayley tables? You're ready. Here's something most people never learn:

Mathematicians don't actually like subtraction and division.

Remember: to solve equations, we need operations where ORDER and GROUPING don't matter.

Test 1: Does order matter? (Commutative Property)

1. Addition: 3 + 5 = _____ 5 + 3 = _____ Same? _____
2. Subtraction: 5 – 3 = _____ 3 – 5 = _____ Same? _____
3. Multiplication: 4 × 6 = _____ 6 × 4 = _____ Same? _____
4. Division: 12 ÷ 3 = _____ 3 ÷ 12 = _____ Same? _____

Test 2: Does grouping matter? (Associative Property)

5. Addition: (2 + 3) + 4 = _____ 2 + (3 + 4) = _____ Same? _____
6. Subtraction: (10 – 5) – 2 = _____ 10 – (5 – 2) = _____ Same? _____
7. Multiplication: (2 × 3) × 4 = _____ 2 × (3 × 4) = _____ Same? _____
8. Division: (24 ÷ 6) ÷ 2 = _____ 24 ÷ (6 ÷ 2) = _____ Same? _____

What did you discover?

9. Which operations KEEP the rules (both commutative AND associative)?

____

■ Discovery Question

10. If subtraction and division break the rules we need, how can mathematicians still "take away" and "split into groups"?

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Part 2: Inventing Negative Numbers

In your color tables, every color had an ADDITIVE INVERSE — a partner that brought it back to Black (the identity for ⊕). What about regular numbers?

The Problem: What number can you add to 5 to get 0? 5 + _____ = 0

With counting numbers (1, 2, 3, 4...), there IS no answer! 5 has no inverse.

The Solution: Mathematicians INVENTED the inverse. They called it –5 ("negative 5").

■ The Money Way to Understand

- Positive numbers = money you HAVE
- Negative numbers = money you OWE
- Your NET WORTH = add them all up

You don't need complicated rules — just figure out your total!

Practice: Figure out the net worth

11. You have $10 and you owe $3.
    10 + (–3) = _____

12. You have $5 and you owe $8.
    5 + (–8) = _____

13. You owe $4 and then owe $6 more.
    (–4) + (–6) = _____

14. You owe $7, but someone forgives $4.
    (–7) + 4 = _____

15. You have $12, owe $5, then get $3 more.
    12 + (–5) + 3 = _____

16. You owe $10, owe $5 more, then get $8.
    (–10) + (–5) + 8 = _____

Every number now has an inverse! Fill in the blanks:

17. 3's additive inverse is _____ (because 3 + _____ = 0)
18. 7's additive inverse is _____ (because 7 + _____ = 0)
19. (–4)'s additive inverse is _____ (because (–4) + _____ = 0)

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Part 3: The Secret About Subtraction

■ The Big Reveal:

Subtraction is secretly just ADDITION OF THE INVERSE!

Instead of subtracting a number, you can ADD its inverse. Same result!

Example: 5 – 3 = 5 + (–3) = 2 ✓ Same answer!

Practice: Rewrite each subtraction as addition, then solve

20. 8 – 3 = 8 + (___) = _____
21. 12 – 7 = 12 + (___) = _____
22. 4 – 9 = 4 + (___) = _____
23. 15 – 15 = 15 + (___) = _____
24. 6 – (–2) = 6 + (___) = _____

■ The Parallel to Color Tables

Color World: To undo ⊕ Yellow, you ⊕ add Yellow's inverse (Orange)

Number World: To undo + 5, you + add 5's inverse (–5)

SAME STRUCTURE! You already knew this pattern!

Solve equations by adding the inverse to both sides:

25. x + 5 = 12

    Add _____ to both sides: x + 5 + (___) = 12 + (___)

    Simplify: x = _____

26. x + 9 = 4

    Add _____ to both sides: x = _____

27. x + (–3) = 10

    Add _____ to both sides: x = _____

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Part 4: Inventing Fractions

Now let's do the same thing for MULTIPLICATION.

The Problem: What number can you multiply by 3 to get 1? 3 × _____ = 1

With whole numbers, there IS no answer! 3 has no multiplicative inverse.

The Solution: Mathematicians INVENTED it. They called it 1/3 ("one third").

A fraction like 1/3 is simply the MULTIPLICATIVE INVERSE of 3.

Find each number's multiplicative inverse:

28. 2's multiplicative inverse is _____ (because 2 × _____ = 1)
29. 5's multiplicative inverse is _____ (because 5 × _____ = 1)
30. 10's multiplicative inverse is _____ (because 10 × _____ = 1)
31. (1/2)'s multiplicative inverse is _____ (because 1/2 × _____ = 1)

Part 5: The Secret About Division

■ The Big Reveal:

Division is secretly just MULTIPLICATION BY THE INVERSE!

Instead of dividing by a number, you can MULTIPLY by its inverse. Same result!

Example: 6 ÷ 2 = 6 × (1/2) = 3 ✓ Same answer!

Practice: Rewrite each division as multiplication, then solve

32. 12 ÷ 4 = 12 × (___) = _____
33. 20 ÷ 5 = 20 × (___) = _____
34. 15 ÷ 3 = 15 × (___) = _____
35. 8 ÷ (1/2) = 8 × (___) = _____ (tricky!)

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Part 6: The Complete Picture — Same Structure!

Fill in this table to see the parallel between color operations and number operations:

	Color World	Number World
36-41. "Add" operation	⊕	+
36-42. "Add" identity	Black	____
36-43. "Multiply" operation	★	×
36-44. "Multiply" identity	White	____
Yellow's add inverse	Orange	2's add inverse = ____
Green's mult inverse	Green	5's mult inverse = ____

Solving equations — same method in both worlds:

Color World

Solve: ? ⊕ Yellow = Green

Add Yellow's inverse (Orange)

? = Orange ⊕ Green = White

Number World

Solve: $x + 3 = 7$

Add 3's inverse (–3)

$x = 7 + (-3) = 4$

■ The Big Idea

You don't have four operations — you have TWO:

• Addition (and its inverse, which we write as subtraction)
• Multiplication (and its inverse, which we write as division)

■ Final Discovery Questions

42. In your 6-color table, some colors had NO multiplicative inverse (like Yellow).
With fractions, does every non-zero number have a multiplicative inverse? YES / NO

43. There's ONE number that still has no multiplicative inverse — even with fractions.
Which number? _____ Why? (Hint: what would you need to multiply it by to get 1?)

____

44. In what way is this number like Black in the color ★ table?

____

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■ Bonus Challenges

Mixed Practice: Solve each equation using inverses (no guessing!)

45. $x + 8 = 3$

Add _____ to both sides → x = _____

46. $x - 5 = 12$

(Hint: rewrite as $x + (-5) = 12$) → x = _____

47. $4x = 20$

Multiply both sides by _____ → x = _____

48. $x \div 3 = 6$

(Hint: rewrite as $x \times (1/3) = 6$) → x = _____

49. $-2 + x = 7$

Add _____ to both sides → x = _____

50. $x \times (1/2) = 10$

Multiply both sides by _____ → x = _____

■ Reflection

51. You learned to solve equations with colored circles BEFORE you learned negative numbers and fractions. How did the colors help you understand the structure?